Positive-Definite Energy Density and Global Consequences for General Relativity*

ARTHUR KOMAR*
Syracuse University, Syracuse, New York
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An expression for energy density in general relativity is constructed starting from the viewpoint that such an expression should be represented by the generator of translations of a preferred set of space-like coordinate surfaces. The resulting expression is shown to yield a positive-definite value for the energy. It is further shown that, in conformity with one version of Mach's principle, if the total energy vanishes, the space is locally flat.

I. INTRODUCTION

THE expressions for energy and momentum densities for Lorentz-covariant field theories are readily obtained from a consideration of the generators of rigid translations of the coordinate surfaces of space-time. The energy and momentum densities can be expressed as some components of a Lorentz-covariant stress tensor, and the total energy and momentum as the components of a Lorentz-covariant free vector. The Lorentz transformations are characterized by the property that they leave invariant the Minkowski metric of space-time.

When one turns to the theory of general relativity, one can consider defining energy and momentum in the analogous fashion by considering the generators of the rigid translations of the coordinate surfaces. We are led in this fashion to the construction of the Einstein pseudotensor. However, in view of the fact that the coordinate surfaces of general relativity are arbitrary, what appears to be the generator of a rigid translation in one coordinate system no longer has this property when a coordinate transformation is performed. This has as a consequence that the Einstein pseudotensor is not really a geometric object and therefore does not appear to give a meaningful localization of energy and momentum. For spaces which are spatially asymptotically flat, for which coordinate systems can be found which are asymptotically rectangular, the Lorentz group frequently can be reintroduced in a preferred fashion so that the total energy and momentum can again be combined to form a Lorentz free vector. There is reason to believe that such Lorentz-covariant constructions in general relativity are necessarily constants of the motion,[1] and, therefore, of questionable value for the purpose of discussing the radiation of gravitational energy. There have been many subsequent attempts to construct expressions for energy and momentum densities,[2–4] most of which has essentially the same vices and virtues as the Einstein pseudotensor.

Despite the fact that the localization of energy in the gravitational field has proven to be such a recalcitrant problem, Brill[5] has shown that there is a definite sense in which one can assert that energy may be associated with a pure gravitational field. He has demonstrated that for the special case of source-free, nonsingular, time-symmetric, axially symmetric, asymptotically Schwarzschild gravitational field, the total energy, as measured by the strength of the $1/r$ terms in the asymptotic behavior of the metric, is necessarily positive-definite.

More recently, Arnowitt, Deser, and Misner[6] have shown that the energy of the gravitational field, as identified by the asymptotic behavior of the metric, is positive-definite for the case where, initially, the spatial metric can be made isotropic on the space-like hypersurface which satisfies the coordinate condition that the trace of the second fundamental form vanishes. They also observed that when the energy of such spaces vanishes the metric is locally flat. A result of more general applicability has been obtained by Peres,[7] who has demonstrated that if a coordinate system is chosen so that the hypersurfaces of constant time are minimal, and if the energy is determined by the strength of the $1/r$ in the asymptotic behavior of the g_{00} component of the metric, then the energy is necessarily positive-definite.

The general theory of relativity is invariant under arbitrary curvilinear coordinate transformations. We can, therefore, obtain infinitely many conserved quantities by determining the generators of the arbitrary infinitesimal coordinate transformations:

$$\bar{x}^i = x^i + \xi^i, \tag{1}$$

where ξ^i is an arbitrary vector field. We can obtain from such considerations[8] what we may call the "generalized" energy flux vector:

$$E^i(\xi) = 2(\xi^{i;l} - \xi^{l;i})_{;l}, \tag{2}$$

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[6] R. Arnowitt, S. Deser, and C. W. Misner, Ann. Phys. (N. Y.) 11, 116 (1960).
[7] A. Peres (private communication from C. W. Misner).
[8] A. Komar, Phys. Rev. 113, 934 (1959).

and the conservation of the total "generalized" energy:

$$E(\xi)=\frac{1}{2\kappa}\int E^m dS_m=\frac{1}{2\kappa}\int E^4 d_3x, \qquad (3)$$

where κ is Einstein's gravitational constant. Such an infinity of conserved quantities can also be obtained in flat Minkowski space-times by the introduction of curvilinear coordinate systems. What singles out the preferred conserved quantities of Lorentz-covariant theories is that the descriptor vector fields, ξ^i, describe coordinate transformations which preserve the Minkowski form of the metric tensor for that class of coordinate system in which the metric tensor was initially in the Minkowski form. That is, the descriptors of the preferred coordinate transformations are hypersurface-orthogonal vector fields which satisfy Killing's equation:

$$\xi_{i;j}+\xi_{j;i}=0. \qquad (4)$$

For more general Riemannian manifolds it would appear that should there exist Killing vector fields, they would be the appropriate fields to use in Eqs. (2) and (3) in order to obtain the analogs of the preferred conserved quantities of Lorentz-covariant theories.[9,10]

In general, however, Killing vectors cannot be found in arbitrary curved spaces. We might expect that perhaps some weakening of the Killing condition could be found which would not limit the geometry of the space, but which would also give a preferred set of conserved quantities. From an analysis of the asymptotic properties of radiative solutions[10] it has been suggested that one could consider using semi-Killing vector fields, ξ^i, defined by

$$\begin{aligned}\xi^m(\xi_{m;i}+\xi_{i;m})&=0,\\ \xi^m{}_{;m}&=0,\end{aligned} \qquad (5)$$

for the descriptors of the preferred time translations. If the vector fields are also assumed to be hypersurface-orthogonal, the energy, as determined by Eq. (3), can then be shown [10] to be positive-definite. Although any four of the five conditions of Eq. (5) can be satisfied locally in an arbitrary space-time manifold, the fifth condition represents a real limitation on the geometry of the manifold.

The purpose of the remainder of this paper is to determine a set of conditions which should be imposed upon the descriptor field in order to assure that the "generalized" energy associated with such a preferred vector field appears as a reasonable generalization of the concept of energy found in Lorentz-covariant theories. The criteria which we would like to use in order to determine what should constitute a reasonable generalization are: (a) the condition imposed on the descriptor field should be applicable, at least locally, in arbitrary manifolds, and thus imply no limitation on the local geometry; (b) should Killing or semi-Killing vector fields exist, they should automatically be compatible with the (sought for) condition; (c) the resulting expression for energy as determined by Eqs. (2) and (3) should be positive-definite; (d) when the total energy vanishes, the space should be locally flat. This last requirement may be regarded as an expression of Mach's principle in the sense that it affirms that an empty universe can have no physics.

II. MINIMAL FIELDS

In conformity with the argument which led to the consideration of the semi-Killing vector fields,[10] we select the descriptor field of Eq. (2), ξ^i, to be hypersurface-orthogonal and time-like, and perform the surface integration of Eq. (3) over the space-like hypersurface orthogonal to ξ^i. More specifically, we have for the surface of integration:

$$dS_i\sim(\xi^m\xi_m)^{-1}\xi_i=-|\xi^m\xi_m|^{-1}\xi_i. \qquad (6)$$

[The last step in Eq. (6) stems from the assumption throughout this paper that the signature of the metric is 1, 1, 1−1.] Thus from Eqs. (2), (3), and (6), we are led to consider the integrand:

$$I=-|\xi^m\xi_m|^{-1}\xi_n(\xi^{n;p}-\xi^{p;n})_{;p}. \qquad (7)$$

If we rearrange some terms, integrate by parts, and make use of the commutation relations for covariant differentiation, we find that we can re-express Eq. (7):

$$\begin{aligned}I=-|\xi^p\xi_p|^{-1}\{&-\xi^{m;n}(\xi_{m;n}+\xi_{n;m})-2\xi^m\xi^nR_{mn}\\ &+2(\xi^m{}_{;m})^2+[\xi_n(\xi^{m;n}+\xi^{n;m})-2\xi^n{}_{;n}\xi^m]_{;m}\}.\end{aligned} \qquad (8)$$

Let us define the vector M_i:

$$M_i\equiv\xi^m(\xi_{m;i}+\xi_{i;m})-2\xi^m{}_{;m}\xi_i, \qquad (9)$$

and the positive-definite spatial metric tensor:

$$\gamma_{ij}\equiv g_{ij}-\xi_i\xi_j(\xi^m\xi_m)^{-1}. \qquad (10)$$

We then find that by an elaborate rearrangement of the terms in Eq. (8) we can express I thus:

$$\begin{aligned}I=+|\xi^p\xi_p|^{-1}\{&+\tfrac{1}{2}(\xi_{m;n}+\xi_{n;m})(\xi_{r;s}+\xi_{s;r})\gamma^{mr}\gamma^{ns}\\ &+2\xi^m\xi^nR_{mn}-M^m{}_{;m}+M^m(\xi^r\xi_r)^{-1}[\xi_m\xi^s{}_{;s}\\ &+(\xi_{m;n}+\xi_{n;m})\xi^n-\xi_m\xi^s\xi^t\xi_{s;t}(\xi^q\xi_q)^{-1}]\}.\end{aligned} \qquad (11)$$

The first term within the bracket of Eq. (11) is manifestly positive-definite. The term containing the Ricci tensor is positive-definite for all customary energy-momentum tensors which exclude negative pressure terms. In fact, for the energy-momentum tensors of the usual nongravitational fields:

$$\begin{aligned}\xi^m\xi^nR_{mn}&\geq 0,\\ \xi^m\xi^nR_{mn}=0&\rightleftarrows R_{mn}=0.\end{aligned} \qquad (12)$$

We are, therefore, led to the consideration of the

[9] A. Trautman, Lectures on Relativity, Kings College, London, 1958 (unpublished).

[10] A. Komar, Phys. Rev. **127**, 1411 (1962).

condition:

$$M_i \equiv \xi^m(\xi_{m;i}+\xi_{i;m})-2\xi^m{}_{;m}\xi_i=0. \quad (13)$$

It is evident from Eq. (11) that this is precisely the condition required to obtain a positive-definite expression for the energy. Furthermore, it we note Eqs. (4) and (5), we see that Killing vector fields and semi-Killing vector fields satisfy Eq. (13) identically. Since it was essential to our construction that ξ^i be hypersurface-orthogonal, let us assume that ξ^i has the form:

$$\xi_i=\lambda\phi_{;i}, \quad (14)$$

where λ and ϕ are scalar fields. Substituting Eq. (14) into Eq. (13), we find that ϕ must satisfy the differential equation:

$$\phi^{;m}\phi^{;n}\phi_{;mn}=\phi^{;m}{}_m\phi^{;n}\phi_{;n}, \quad (15)$$

which is the equation of a family of minimal surfaces. Conversely, given a solution of Eq. (15), if we define λ to be

$$\lambda=(\phi^{;m}\phi_{;m})^{-1}, \quad (16)$$

we find that the ξ^i of Eq. (14) is automatically a solution of the condition Eq. (13). The hypersurface-orthogonal vector fields, ξ^i, which satisfy Eq. (13) are, therefore, found to be the orthogonal trajectories of a family of spacelike minimal surfaces. It is, therefore, evident that such a family of descriptors can be found locally in an arbitrary Riemannian manifold, and thus Eq. (13) implies no restriction on the geometry of the space-time. It seems natural to name the ξ^i which satisfy Eq. (13) "minimal vector fields." Our proof that minimal vector fields provide us with preferred descriptors which yield a positive-definite expression for the energy is evidently very closely related to the work of Peres[7] mentioned in the introduction.

III. GLOBAL CONSIDERATIONS

Thus far we have shown that by the employment of a time-like vector field as a preferred descriptor, we obtain an expression for energy which is in conformity with criteria (a), (b), and (c) of Sec. I. We now wish to investigate the validity of criterion (d), namely, the assertion that if a space is empty it is flat.

Let us, therefore, consider the case of

$$E(\xi)=0, \quad (17)$$

where the descriptor ξ^i satisfies Eq. (13). From Eq. (11) we can conclude that, for spaces which have energy-momentum tensors which conform to Eq. (12):

$$R_{ij}=0, \quad (18)$$

$$(\xi_{m;n}+\xi_{n;m})(\xi_{\lambda;s}+\xi_{s;\lambda})\gamma^{m\lambda}\gamma^{ns}=0. \quad (19)$$

In view of the positive-definite character of γ^{mn}, as defined by Eq. (10), it follows from Eq. (19) that

$$\xi_{i;j}+\xi_{j;i}-\xi_i(\xi^p\xi_p)^{-1}\xi^m(\xi_{m;j}+\xi_{j;m}) -\xi_j(\xi^p\xi_p)^{-1}\xi^m(\xi_{m;i}+\xi_{i;m}) +2\xi_i\xi_j(\xi^p\xi_p)^{-2}\xi^m\xi^n\xi_{m;n}=0. \quad (20)$$

Thus combining Eq. (13) and (20) we see that ξ^i must satisfy the equation

$$\xi_{i;j}+\xi_{j;i}=2\xi_i\xi_j(\xi^p\xi_p)^{-1}\xi^m{}_{;m}. \quad (21)$$

A vector field which satisfied Eq. (21) can be called almost-Killing in view of the fact that it satisfies all but one of the Killing equations. If a coordinate system chosen so that a hypersurface-orthogonal almost-Killing vector field, ξ^i, has the form

$$\xi^i=\delta_4{}^i, \quad (22)$$

the metric tensor in such a coordinate patch will be found to have the form

$$g_{\alpha\beta,4}=g_{\alpha 4,4}=0. \quad (23)$$

(Greek indices are assumed to run from 1 to 3). We should note that $g_{44,4}$ is arbitrary, which fact distinguishes the almost-Killing vector fields from the Killing vector fields. If we employ the coordinate system exemplified by Eqs. (22) and (23) (or, equivalently, use the method of exterior differential forms adapted to the vector field ξ^i), we find

$$\xi^m\xi^n R_{mn}=-\xi\xi_{:\alpha}{}^{\alpha}, \quad (24)$$

where the scalar ξ is defined by

$$\xi=(\xi^p\xi_p)^{1/2}, \quad (25)$$

and the colon is used to denote covariant differentiation with respect to the spatial metric $\gamma_{\alpha\beta}$. Greek indices are also raised and lowered by means of the spatial metric. Equation (24) is identical to the one obtainable under the assumption that ξ^i is Killing, and the remainder of the discussion may proceed as in the latter case.[11] More concisely, it is sufficient for our present purposes to note that in view of Eq. (18) we have from Eq. (24)

$$\nabla^2\xi\equiv\xi^{:\alpha}{}_{\alpha}=0. \quad (26)$$

Since the spatial metric is positive-definite, the only nonsingular solution of Eq. (26) (subject to the usual boundary conditions of either a spatially compact or almost periodic manifold, or ξ approaching a finite constant value at infinity for an asymptotically flat manifold) is

$$\xi=\text{const.} \quad (27)$$

Differentiating Eq. (27) we find

$$\xi^m\xi_{m;i}=0, \quad (28)$$

and therefore by taking the scalar product of Eq. (13) with ξ^i we have

$$\xi^m{}_{;m}=0. \quad (29)$$

We can now conclude from Eq. (21) that ξ^i satisfies Eq. (4) and is, therefore, Killing. We may now refer to a well-known global theorem[11] and assert that subject to the previously mentioned boundary conditions the

[11] A. Lichnerowicz, *Theories Relativistes de la Gravitation et de l'Electromagnetisme* (Masson et Cie, Paris, 1955).

space is locally flat. Our expression for energy is therefore in conformity with criterion (d) as well.

IV. CONCLUSION

Starting from the viewpoint that the expression for energy should be represented by the generator of translations of a preferred set of space-like coordinate surfaces, we were led to the introduction of the minimal vector fields as preferred descriptors. The expression for energy density so obtained has the virtue of being generally applicable, of yielding a positive-definite value for the energy, of corresponding to the preferred energy expressions of Lorentz-covariant theories, and of being in conformity with Mach's principle to the extent that we may conclude that empty spaces (i.e., those for which the total energy vanishes) are locally flat.

The energy density constructed in this paper still has one essential defect in our opinion. In a local neighborhood there is a high degree of arbitrariness in the construction of families of space-like minimal surfaces, and a consequent high degree of arbitrariness in the definition of our energy density. We can attempt to reduce this arbitrariness by employing appropriate boundary conditions. In fact, we know from the work of reference 10 that for asymptotically flat space-times it is essential that the descriptor fields be asymptotically semi-Killing if our "generalized energy" is to be well defined, and is to coincide in value with the usual expression for the mass of asymptotically Schwarzschild solutions. However, it is evident that such attempts to reduce the arbitrariness of the energy expression lead to definitions which are necessarily nonlocal. Perhaps this is the best that one can hope for, but certainly a reasonably unique local definition of energy density would have been preferable.

For spatially closed space-times it is evident from Eqs. (2) and (3) that the total generalized energy vanishes. We thus see that global families of closed minimal surfaces with $\beta_2=0$ can only be found for the (trivial) locally flat space-times. This may be regarded by some as a serious drawback of our construction. An expression suitable for spatially closed space-times will be developed in a subsequent paper.

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Regge Trajectory in Field Theory*

LU SUN LIU† AND K. TANAKA
Argonne National Laboratory, Argonne, Illinois
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The form of the Regge trajectory, the graph of the complex angular momentum $\alpha(t)$ as a function of the square of the momentum transfer t, is studied in field theory within the framework of the ladder approximation in the crossed channel. To order g^2, the trajectory is unbounded in the region $t\approx 4m^2$, but inclusion of the higher-order terms in g^2 removes the divergence and leads to a smooth trajectory that resembles that expected from potential scattering. The various relations between g^2, $\alpha(t)$, and the exchanged mass are discussed.

1. INTRODUCTION

MANY attempts have been made to correlate high-energy experimental data in terms of Regge poles which are generalized bound states and resonances in the complex angular-momentum plane.[1] There is no rigorous proof of the existence of Regge poles in relativistic field thory. Therefore, information on the possible trajectory [in particular, the energy dependence of the complex angular momentum $\alpha(t)$ in field theory] is inferred from potential scattering, where the existence of Regge poles rests on a secure foundation.[2]

Several authors[3,4] have demonstrated that it is possible to obtain some information on the Regge trajectory of $\alpha(t)$ in field theory, within the framework of the ladder approximation in the crossed channel.

The purpose of this paper is to re-examine the possible Regge trajectory in field theory, within the ladder approximation. A comparison of the present approach with perturbation theory shows that the higher-order terms in the coupling constant make very important contributions to the trajectory. A simple derivation of the trajectory equation is given in Sec. 2. The trajectory is discussed in Sec. 3, the case $\alpha(0)$ is treated in Sec. 4, and finally a summary is given in Sec. 5.

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† Also at Northwestern University, Evanston, Illinois.

[1] See, for example, G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961) and **8**, 41 (1962).

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[3] L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento **25**, 626 (1962), hereafter called BFT. This contains references to earlier work.

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